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PERTURBATION THEORY FOR LYAPUNOV EXPONENTS OF A TORAL MAP: EXTENSION OF A RESULT OF SHUB AND WILKINSON

BY

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ABSTRACT

Starting from a hyperbolic toral automorphism times a rotation of the circle, we obtain, for a small volume preserving perturbation, an exact and rigorous second order perturbation expansion of the Lyapunov exponents.

Introduction

We consider volume preserving perturbations F of a diffeomorphism $F_0 = (\Phi, J)$ of $\mathbf{T}^{m+1} = \mathbf{T}^m \times \mathbf{T}$, where Φ is a hyperbolic automorphism of $\mathbf{T}^m$, and J is a translation of $\mathbf{T}$. Writing $F = F_0 + aF'$, we shall show that the Lyapunov exponents for (F, volume) can be expanded to second order in a (Theorem 1). In particular, the central Lyapunov exponent λ^c of (F, volume), to second order in a, is generally $\neq 0$ (Corollary 11). For a special family of perturbations one obtains particularly simple formulae, first noted by Shub and Wilkinson [17]. We recover their result in Theorem 12. We deviate from [17] mostly in that we don't have differentiability of λ^c, only a second order expansion around $a = 0$. The ideas used here are largely those in Shub and Wilkinson [17], and can be appreciated in the background provided by Hirsh, Pugh and Shub [9], Burns and Wilkinson [5], Ruelle and Wilkinson [16], Niţică and Török [12], Pugh, Shub and Wilkinson

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[14]. Among older regularity results let us mention Katok, Knieper and Weiss [11], Flaminio [9], Ruelle [15]. For recent work concerning Lyapunov exponents, see Bonatti, Gómez-Mont and Viana [3], Avila and Bochi [2]. Closely related to the subject of the present paper are the references [4] and [6].

After completing the writing of this paper, the author received a preprint by D. Dolgopyat [7], which develops similar ideas in a more general setting, but without the specific formulas we obtain here.

1. Theorem

Let Φ be a hyperbolic automorphism of $\mathbf{T}^m$, and J: $y \mapsto y + \alpha \pmod 1$ a translation of $\mathbf{T}$. Define $F_0 = (\Phi, J)$, and let $F = F_0 + aF'$(+higher order in a) be a C^2 perturbation of F_0, volume preserving to first order in a. (We take F': $\mathbf{T}^{m+1} \mapsto \mathbf{R}^{m+1}$ and $F_0\xi + aF'(\xi)$ has to be understood (mod 1) in each component.) Let $\lambda_1 < \lambda_2 < \cdots$ be the Lyapunov exponents of (F_0, volume) and $m_1, m_2, \ldots$ their multiplicities (the exponent $= 0$ occurs with multiplicity 1). Also let $\lambda_a^{(1)} \leq \lambda_a^{(2)} \leq \cdots$ be the Lyapunov exponents of (F, volume) repeated according to multiplicity. Then we have the second order expansion

$$\sum_{\ell=m_1+\cdots+m_{r-1}+1}^{m_1+\cdots+m_r} \lambda_a^{(\ell)} = m_r\lambda_r + a^2 L_r + o(a^2).$$

If $m_r = 1$, and writing $\lambda_r = \lambda_0^{(\ell)}$, we have

$$\lambda_a^{(\ell)} = \lambda_0^{(\ell)} + a^2 L^{(\ell)} + o(a^2)$$

(this applies in particular to $\lambda^c = \lambda_a^{(\ell)}$ for $\lambda_0^{(\ell)} = 0$).

An explicit expression for L_r can be obtained (see Proposition 9). We do not assume ergodicity of (F, volume), and therefore we use *integrated* Lyapunov exponents (averaged over the volume); see, however, Remark 15(a).

Because the perturbation $+aF'$(+higher order in a) to F_0 gives only a quadratic contribution in the above formulas, the higher order terms do not contribute to order a^2. Since the higher order terms do not change our results, these terms will be omitted in what follows.

2. Normal hyperbolicity

As in [17], we invoke the theory of normal hyperbolicity of [10]. We start from the fact that F_0 is normally hyperbolic to the smooth fibration of $\mathbf{T}^{m+1}$ by circles

$\{x\} \times \mathbf{T}$. Taking some $k \geq 2$ we apply [10] Theorems (7.1), (7.2). Thus we obtain a C^1 neighborhood U of F_0 in the C^k diffeomorphisms of $\mathbf{T}^{m+1}$ such that, for $F \in U$, there is an equivariant fibration π: $\mathbf{T}^{m+1} \to \mathbf{T}^m$ with

$$\pi F = \Phi\pi.$$

The fibers $\pi^{-1}\{x\}$ are C^k circles forming a continuous fibration of $\mathbf{T}^{m+1}$ (this fibration is in general not smooth). Furthermore, there is a TF-invariant continuous splitting of $T\mathbf{T}^{m+1}$ into three subbundles:

$$T\mathbf{T}^{m+1} = E^s + E^u + E^c$$

such that E^c is 1-dimensional tangent to the circles $\pi^{-1}\{x\}$, E^s is m^s-dimensional contracting and E^u is m^u-dimensional expanding for TF.

If $\lambda_r < 0$ (and F is in a suitable C^1-small neighborhood U of F_0), we can introduce a continuous vector subbundle E^r of $T\mathbf{T}^{m+1}$ which consists of vectors contracting under TF^n faster than $(\lambda_r + \epsilon)^n$ where $\epsilon > 0$ and $\lambda_r + \epsilon < \lambda_{r+1}$. In fact E^r is a hyperbolic (attracting) fixed point for the action induced by TF^{-1} on the bundle of $m_1 + \cdots + m_r$ dimensional linear subspaces of $T\mathbf{T}^{m+1}$ (over F^{-1} acting on T^{m+1}).

If $\lambda_r > 0$, replacement of F by F^{-1} similarly yields a continuous subbundle $\bar{E}^r$ of $m_r + \cdots$ dimensional subspaces.

3. Proposition

Assume that F is of class C^k, $k \geq 2$, and that F is C^k close to F_0. The bundles E^r, $\bar{E}^r$ when restricted to a circle $\pi^{-1}\{x\}$ are of class C^{k-1}, continuously in x.

If $\mathcal{G}$ denotes the (Grassmannian) manifold of $m_1 + \cdots + m_r$ dimensional linear subspaces of $\mathbf{R}^{m+1}$, we may identify the bundle of $m_1 + \cdots + m_r$ dimensional linear subspaces of $T\mathbf{T}^{m+1}$ with $\mathbf{T}^{m+1} \times \mathcal{G}$. We denote by $\mathcal{E} \in \mathcal{G}$ the spectral subspace of the matrix defining Φ corresponding to the smallest $m_1 + \cdots + m_r$ eigenvalues (in absolute value, and repeated according to multiplicity).

If $\mathcal{F}_0$ is the action defined by TF_0 on $T\mathbf{T}^{m+1} \times \mathcal{G}$, the circles $\{x\} \times \mathbf{T} \times \{\mathcal{E}\}$ form an $\mathcal{F}_0$ invariant fibration of $\mathbf{T}^{m+1} \times \{\mathcal{E}\}$, to which $\mathcal{F}_0$ is normally hyperbolic. If F is C^k close to F_0, the corresponding C^{k-1} action $\mathcal{F}$ is normally hyberbolic to a pertubed fibration where $\{x\} \times \mathbf{T} \times \{\mathcal{E}\}$ is replaced by $E^r|\pi^{-1}\{x\}$. According to [10] Theorem 7.4, Corollary (8.3) and the following Remark 2, we find that the C^{k-1} circle $E^r|\pi^{-1}\{x\} \subset \mathbf{T}^{m+1} \times \mathcal{G}$ depends continuously on $x \in \mathbf{T}^{m+1}$. Similarly for $\bar{E}$. ∎

Note that in [17], the C^r section theorem is used in a similar situation, giving estimates uniform in x. However, continuity in x (not just uniformity) will be essential for us in what follows.

4. Corollary

The splitting $T\mathbf{T}^{m+1} = E^s + E^u + E^c$ *when restricted to a circle* $\pi^{-1}\{x\}$ *is of class* C^{k-1}*, continuously in* x.

It is clear that $E^c|\pi^{-1}\{x\}$ is of class C^{k-1} because it is the tangent bundle to the C^k circle $\pi^{-1}\{x\}$. As to E^s, E^u, they are special cases of E^r, $\bar{E}^r$. ∎

Notation: Remember that $F = F_0 + aF'$, and fix F'. We shall use the notation π_a, E^r_a, ... to indicate the a-dependence of π, E^r, ...

5. Proposition

For small $\epsilon > 0$ *there is a continuous function* $x \mapsto \gamma_x$ *from* $\mathbf{T}^m$ *to* $C^k(\mathbf{T} \times (-\epsilon, \epsilon) \to \mathbf{T}^m)$ *such that* $\gamma_x(y, 0) = 0$ *and*

$$\pi_a^{-1}\{x\} = \{(x + \gamma_x(y, a), y) \colon y \in \mathbf{T}\}.$$

To see this define $\tilde{F}\colon \mathbf{T}^{m+1} \times (-\epsilon, \epsilon) \to \mathbf{T}^{m+1} \times (-\epsilon, \epsilon)$ by

$$\tilde{F}(\xi, a) = ((F_0 + aF')(\xi), a)$$

and observe that $\tilde{F}$ is normally hyperbolic to the 2-dimensional manifolds

$$\bigcup_{a \in (-\epsilon, \epsilon)} (\pi_a^{-1}\{x\}, a)$$

and these are thus C^k 2-dimensional submanifolds of $\mathbf{T}^{m+1} \times (-\epsilon, \epsilon)$. ∎

We may in the same manner replace $\pi_a^{-1}\{x\}$ by $\bigcup_{a \in (-\epsilon,\epsilon)} (\pi_a^{-1}\{x\}, a)$ in Proposition 3 and Corollary 4. Writing E_a for E^r_a, $\bar{E}^r_a$, E^s_a, E^u_a, E^c_a, we obtain that $(\cdot, a) \mapsto E_a(\cdot)$, when restricted from $\mathbf{T}^{m+1} \times (-\epsilon, \epsilon)$ to $\bigcup_{a \in (-\epsilon,\epsilon)} (\pi_a^{-1}\{x\}, a)$, is of class C^{k-1}. We rephrase this as follows:

6. Proposition

The map

$$x \mapsto \{(y,a) \mapsto E_a(x+\gamma_x(y,a),y)\},$$

where E_a *stands for* E_a^r, $\bar{E}_a^r$, E_a^s, E_a^u, E_a^c, *is continuous* $\mathbf{T}^m \to C^{k-1}(\mathbf{T}\times(-\epsilon,\epsilon) \to$ *Grassmannian of* $\mathbf{R}^{m+1})$ *where we have used the identification* $T\mathbf{T}^{m+1} = \mathbf{T}^{m+1} \times \mathbf{R}^{m+1}$.

Notation: From now on we write E_a for E_a^r, $\bar{E}_a^r$, E_a^s, E_a^u, E_a^c. When $a=0$, E_0 is a trivial subbundle of $T\mathbf{T}^{m+1} = \mathbf{T}^{m+1} \times \mathbf{R}^{m+1}$, and we shall write $E_0 = \mathbf{T}^{m+1} \times \mathcal{E}$, denoting thus by $\mathcal{E}$ a spectral subspace of the matrix on $\mathbf{R}^{m+1}$ defining $(\Phi, 1)$. We denote by $\mathcal{E}^\perp$ the complementary spectral subspace.

Taking $k=2$ we have then:

7. Corollary

There are linear maps $G(x,y), R(x,y,a)\colon \mathcal{E} \to \mathcal{E}^\perp$ *such that* $G(x,y)$ *depends continuously on* $(x,y) \in \mathbf{T}^m \times \mathbf{T}$, $R(x,y,a)$ *on* $(x,y,a) \in \mathbf{T}^m \times \mathbf{T} \times (-\epsilon,\epsilon)$,

$$E_a(x+\gamma_x(y,a),y) = \{X + aG(x,y)X + R(x,y,a)X \colon X \in \mathcal{E}\}$$

and $||R(x,y,a)||$ *is* $o(a)$ *uniformly in* x, y.

Notice now that, if $\tilde{x} = \pi_a(x,y)$, then $x = \tilde{x} + \gamma_{\tilde{x}}(y,a)$, where $\gamma_{\tilde{x}}(y,a) = O(a)$. Now

$$E_a(x,y) = E_a(\tilde{x}+\gamma_{\tilde{x}}(y,a),y) = \{X + aG(\tilde{x},y)X + R(\tilde{x},y,a)X \colon X \in \mathcal{E}\}$$

differs from

$$E_a(x+\gamma_x(y,a),y) = \{X + aG(x,y)X + R(x,y,a)X \colon X \in \mathcal{E}\}$$

by the replacement $\tilde{x} \to x$ in the right-hand side, and since $\mathrm{dist}(\tilde{x},x) = O(a)$, we find that $\mathrm{dist}(E_a(x,y), E_a(x+\gamma_x(y,a),y)) = o(a)$. Therefore, changing the definition of R, we can again write:

8. Corollary

There are linear maps $G(x,y), R(x,y,a)\colon \mathcal{E} \to \mathcal{E}^\perp$, *depending continuously on their arguments, such that*

$$E_a(x,y) = \{X + aG(x,y)X + R(x,y,a)X \colon X \in \mathcal{E}\}$$

and $||R(x,y,a)||$ is $o(a)$ uniformly in x, y.

We may write $T_\xi F = T_\xi(F_0 + aF') = D_0 + aD'(\xi)$, where D_0 does not depend on ξ and preserves the decomposition $T_\xi M = \mathcal{E} + \mathcal{E}^\perp$. If we apply TF to an element $X + aGX + RX$ of E_a (as in Corollary 8) we obtain X_1+ element of $\mathcal{E}^\perp \in E_a$, with $X_1 \in \mathcal{E}$:

$$X_1 = D_0X + aD'X + a^2D'GX + aD'RX \quad \text{projected on } \mathcal{E}. \tag{1}$$

Under $(TF)^\wedge$, the volume element θ in $E_a(\xi)$ is multiplied by a factor $M(\xi, a)$, and the projection in $\mathcal{E}$ of $(TF)^\wedge\theta$ is equal to the projection in $\mathcal{E}$ of θ multiplied by a factor $N(\xi, a)$ such that

$$M(\xi,a) = N(\xi,a) + \ell_a(\xi) - \ell_a(F\xi)$$

for suitable ℓ_a. We may compute N from (1):

$$N(\xi,a) = N_{(0)} + aN_{(1)}(\xi) + a^2N_{(2)}(\xi) + o(a^2).$$

To proceed we take now $E_a = E_a^r$, and assume $\lambda_r < 0$. We have then, writing $d\xi$ for the volume element in T^{m+1},

$$\begin{aligned} L_a = \sum_{\ell=1}^{m_1+\ldots+m_r} \lambda_a^{(\ell)} &= \int d\xi \, \log M(\xi,a) = \int d\xi \, \log N(\xi,a) \\ &= L_{(0)} + aL_{(1)}(\xi) + a^2L_{(2)}(\xi) + o(a^2). \end{aligned} \tag{2}$$

More precisely, we shall prove

9. Proposition

If $\lambda_r < 0$, we have

$$\sum_{\ell=1}^{m_1+\cdots+m_r} \lambda_a^{(\ell)} = \sum_{k=1}^{r} m_k\lambda_k + a^2L + o(a^2)$$

where

$$L = \frac{1}{2}\sum_{n=-\infty}^{\infty} \int d\xi \, \mathrm{Tr}_{\mathcal{E}}(D_0^{-1}D'(\xi)) \, \mathrm{Tr}_{\mathcal{E}}(D_0^{-1}D'(F_0^n\xi)) \geq 0$$

and $\mathrm{Tr}_{\mathcal{E}}$ is defined as follows. Let $\mathcal{E}$ be the spectral subspace of the matrix D_0 (defining $(\Phi, 1)$ in $\mathbf{R}^{m+1}$) corresponding to the smallest $m_1 + \cdots + m_r$ eigenvalues (in absolute value, and repeated according to multiplicity). Also let $\mathcal{E}^\perp$ be the *complementary spectral subspace. We define P to be the projection on $\mathcal{E}$ parallel to $\mathcal{E}^\perp$, and write* $\mathrm{Tr}_{\mathcal{E}} \cdots = \mathrm{Tr}_{\mathbf{R}^{m+1}} P \cdots P$.

The convergence of the series defining L is exponential, as will result from the proof. We postpone showing that $L \geq 0$ until Remark 15(b).

The proposition is obtained by comparing formula (2) with the formula (5) below, which we shall obtain by a second order perturbation calculation.

To first order in a we have

$$F^n = (F_0 + aF')^n = F_0^n + a \sum_{j=1}^{n} F_0^{n-j} \circ F' \circ F_0^{j-1},$$

hence

$$T_\xi F^n = D_0^n + a \sum_{j=1}^{n} D_0^{n-j} D'(F^{j-1}\xi) D_0^{j-1}.$$

If we apply TF^n to $X + aGX + RX \in E_a$ we obtain $X_n +$ element of $\mathcal{E}^\perp \in E_a$, with $X_n \in \mathcal{E}$. To zeroth order in a, $X_n = D_0^n X$, so we may write to first order $X_n = D_0^n X + aY_n(\xi)$. Therefore, to first order in a,

$$\begin{aligned} D_0^n X + aY_n(\xi) + aG(F^n\xi)D_0^n X =& D_0^n X + a \sum_{j=1}^{n} D_0^{n-j} D'(F^{j-1}\xi) D_0^{j-1} X \\ &+ aD_0^n G(\xi) X \end{aligned}$$

and, taking the components along $\mathcal{E}^\perp$,

$$G(F^n\xi)D_0^n X = \sum_{j=1}^{n} D_0^{n-j} D'_\perp(F^{j-1}\xi) D_0^{j-1} X + D_0^n G(\xi) X,$$

where $D'_\perp(.)$ is $D'(.)$ followed by taking the component along $\mathcal{E}^\perp$, or

$$\sum_{j=1}^{n} D_0^{-j} D'_\perp(F^{j-1}\xi) D_0^{j-1} X + G(\xi) X = D_0^{-n} G(F^n\xi) D_0^n X.$$

When $n \to \infty$, the right-hand side tends to zero (exponentially fast, remember that $X \in \mathcal{E}$, $GX \in \mathcal{E}^\perp$). Therefore (to order 0 in a)

$$G(\xi)X = -\sum_{j=1}^{\infty} D_0^{-j} D'_\perp(F^{j-1}\xi) D_0^{j-1} X,$$

which we shall use in the form

$$G(\xi)X = -\sum_{n=0}^{\infty} D_0^{-n-1} D'_\perp(F_0^n\xi) D_0^n X, \tag{3}$$

where we have written F_0^n instead of F^n since G is evaluated to order 0 in a. (The right-hand side is an exponentially convergent series.)

Returning to (1) we see that, to second order in a,

$$\begin{aligned} X_1 &= D_0 X + aD'(\xi)X + a^2 D'(\xi)G(\xi)X \quad \text{projected on } \mathcal{E}, \\ &= D_0(1 + aD_0^{-1}D'(\xi) + a^2 D_0^{-1}D'(\xi)G(\xi))X \quad \text{projected on } \mathcal{E}. \end{aligned}$$

Let now $(u^{(i)})$ and $(u^{(i)\perp})$ be conjugate bases of $\mathcal{E}$. Also let $\delta^{(i)}$ for $i = 1, \ldots, m_1 + \cdots + m_r$ be the eigenvalues of D_0 restricted to $\mathcal{E}$. Then, to second order in a,

$$N(\xi, a) \wedge_1^{m_1+\cdots+m_r} u^{(\ell)}$$

is, up to a factor of absolute value 1,

$$\begin{aligned} \left(\prod_{\ell=1}^{m_1+\cdots+m_r} \delta^{(\ell)}\right) \Big[1 + a \sum_{i=1}^{m_1+\cdots+m_r} & (u^{(i)\perp}, D_0^{-1}D'(\xi)u^{(i)}) \\ & + a^2 \sum_{i<j} ((u^{(i)\perp}, D_0^{-1}D'(\xi)u^{(i)})(u^{(j)\perp}, D_0^{-1}D'(\xi)u^{(j)}) \\ & - (u^{(i)\perp}, D_0^{-1}D'(\xi)u^{(j)})(u^{(j)\perp}, D_0^{-1}D'(\xi)u^{(i)})) \\ & + a^2 \sum_i (u^{(i)\perp}, D_0^{-1}D'(\xi)G(\xi)u^{(i)}) \Big] \wedge_\ell u^{(\ell)}, \end{aligned}$$

so that

$$\begin{aligned} N(\xi, a) = \left(\prod_{\ell=1}^{m_1+\cdots+m_r} |\delta^{(\ell)}|\right) \Big[1 + \Big\{ & a \sum_i (u^{(i)\perp}, D_0^{-1}D'(\xi)u^{(i)}) \\ & + a^2 \sum_{i<j} ((u^{(i)\perp}, D_0^{-1}D'(\xi)u^{(i)})(u^{(j)\perp}, D_0^{-1}D'(\xi)u^{(j)}) \\ & - (u^{(i)\perp}, D_0^{-1}D'(\xi)u^{(j)})(u^{(j)\perp}, D_0^{-1}D'(\xi)u^{(i)})) \\ & + a^2 \sum_i (u^{(i)\perp}, D_0^{-1}D'(\xi)G(\xi)u^{(i)}) \Big\} \Big]. \end{aligned}$$

Since $\log|\delta^{(\ell)}| = \lambda_0^{(\ell)}$ we obtain, to second order in a,

$$\begin{aligned} L_a &= \int d\xi \log N(\xi, a) \\ &= m_1\lambda_1 + \cdots + m_r\lambda_r + \int d\xi \left[\{\cdots\} - \frac{a^2}{2} \left(\sum_i (u^{(i)\perp}, D_0^{-1}D'(\xi)u^{(i)}) \right)^2 \right], \end{aligned}$$

where $\{\cdots\}$ has the same meaning as above. Write

$$\Psi_i\left(\sum_\ell \xi_\ell u^{(\ell)}\right) = \left(u^{(i)\perp}, D_0^{-1}F'\left(\sum_\ell \xi_\ell u^{(\ell)}\right)\right).$$

The first term of $\int d\xi\{\cdots\}$ is

$$a\sum_i\int d\xi(u^{(i)\perp},D_0^{-1}TF'(\xi)u^{(i)})=a\sum_i\int d\xi\frac{\partial}{\partial\xi_i}\Psi_i,$$

which vanishes because $\int d\xi\frac{\partial}{\partial\xi_i}\cdots=0$. The next term in $\int d\xi\{\cdots\}$ is

$$\begin{aligned}a^2\sum_{i<j}\int d\xi\Big(\Big(\frac{\partial\Psi_i}{\partial\xi_i}\Big)\Big(\frac{\partial\Psi_j}{\partial\xi_j}\Big)-\Big(\frac{\partial\Psi_i}{\partial\xi_j}\Big)\Big(\frac{\partial\Psi_j}{\partial\xi_i}\Big)\Big)&\\ =a^2\sum_{i<j}\int d\xi\Big(\frac{\partial}{\partial\xi_i}\Big(\Psi_i\frac{\partial\Psi_j}{\partial\xi_j}\Big)-\frac{\partial}{\partial\xi_j}\Big(\Psi_i\frac{\partial\Psi_j}{\partial\xi_i}\Big)\Big),&\end{aligned}$$

which vanishes as above. Thus we are left with

$$\begin{aligned}&L_a-(m_1\lambda_1+\cdots+m_r\lambda_r)\\ &=a^2\int d\xi\Big[\sum_i(u^{(i)\perp},D_0^{-1}D'(\xi)G(\xi)u^{(i)})-\frac{1}{2}\Big(\sum_i(u^{(i)\perp},D_0^{-1}D'(\xi)u^{(i)})\Big)^2\Big]\end{aligned}\tag{4}$$

and we may write, using (3),

$$\begin{aligned}&\sum_i(u^{(i)\perp},D_0^{-1}D'(\xi)G(\xi)u^{(i)})\\ &\quad=-\sum_{n=0}^{\infty}\sum_i(u^{(i)\perp},D_0^{-1}D'(\xi)D_0^{-n-1}D'_\perp(F_0^n)D_0^nu^{(i)})\\ &\quad=-\sum_{n=0}^{\infty}\sum_i\sum_j^{*}(u^{(i)\perp},D_0^{-1}D'(\xi)u^{(j)})(u^{(j)\perp},D_0^{-n-1}D'(F_0^n\xi)D_0^nu^{(i)}),\end{aligned}$$

where we have introduced conjugate bases $(u^{(j)})$, $(u^{(j)\perp})$ of $\mathcal{E}^\perp$, indexed by $j=m_1+\cdots+m_r+1,\ldots,m+1$, and $\sum_i$ is over $i\le m_1+\cdots+m_r$, $\sum_j^*$ is over $j\ge m_1+\cdots+m_r+1$. The above expression is also

$$\begin{aligned}=-\sum_{n=0}^{\infty}\sum_i\sum_j^{*}\frac{\partial}{\partial\xi_j}\Big(u^{(i)\perp},D_0^{-1}F'\Big(\sum_\ell\xi_\ell u^{(\ell)}\Big)\Big)&\\ \times\frac{\partial}{\partial\xi_i}\Big(u^{(j)\perp},D_0^{-n-1}F'\Big(F_0^n\sum_\ell\xi_\ell u^{(\ell)}\Big)\Big)&\end{aligned}$$

and integration by parts thus gives

$$\int d\xi\sum_i(u^{(i)\perp},D_0^{-1}D'(\xi)G(\xi)u^{(i)})$$

$$= -\sum_{n=0}^{\infty} \int d\xi \sum_i \frac{\partial}{\partial \xi_i} \Big(u^{(i)\perp}, D_0^{-1} F'\Big(\sum_\ell \xi_\ell u^{(\ell)}\Big)\Big)$$

$$\times \sum_j^* \frac{\partial}{\partial \xi_j} \Big(u^{(j)\perp}, D_0^{-n-1} F'\Big(F_0^n \sum_\ell \xi_\ell u^{(\ell)}\Big)\Big)$$

$$= -\sum_{n=0}^{\infty} \int d\xi \, \mathrm{Tr}_{\mathcal{E}}(D_0^{-1} D'(\xi)) \, \mathrm{Tr}_{\mathcal{E}^\perp}(D_0^{-n-1} D'(F_0^n \xi) D_0^n)$$

$$= -\sum_{n=0}^{\infty} \int d\xi \, \mathrm{Tr}_{\mathcal{E}}(D_0^{-1} D'(\xi)) \, \mathrm{Tr}_{\mathcal{E}^\perp}(D_0^{-1} D'(F_0^n \xi))$$

(here $\mathrm{Tr}_{\mathcal{E}^\perp} = \mathrm{Tr}_{\mathbf{R}^{m+1}} - \mathrm{Tr}_{\mathcal{E}}$). The fact that $F = F_0 + aF'$ is volume preserving (to first order in a) is expressed by $\mathrm{Tr}_{\mathbf{R}^{m+1}}(D_0^{-1} D'(\xi)) = 0$, hence

$$\int d\xi \sum_i (u^{(i)\perp}, D_0^{-1} D'(\xi) G(\xi) u^{(i)})$$

$$= \sum_{n=0}^{\infty} \int d\xi \, \mathrm{Tr}_{\mathcal{E}}(D_0^{-1} D'(\xi)) \, \mathrm{Tr}_{\mathcal{E}}(D_0^{-1} D'(F_0^n \xi)),$$

and introducing this in (4) yields

$$L_a - (m_1\lambda_1 + \cdots + m_r\lambda_r) = a^2 \Big[\sum_{n=1}^{\infty} \int d\xi \, \mathrm{Tr}_{\mathcal{E}}(D_0^{-1} D'(\xi)) \, \mathrm{Tr}_{\mathcal{E}}(D_0^{-1} D'(F_0^n \xi))$$

$$+ \frac{1}{2} \int d\xi (\mathrm{Tr}_{\mathcal{E}}(D_0^{-1} D'(\xi)))^2 \Big]$$

$$= \frac{a^2}{2} \sum_{n=-\infty}^{\infty} \int d\xi \, \mathrm{Tr}_{\mathcal{E}}(D_0^{-1} D'(\xi)) \, \mathrm{Tr}_{\mathcal{E}}(D_0^{-1} D'(F_0^n \xi)), \tag{5}$$

where the last step used the invariance of $d\xi$ under F_0^n. ∎

10. Proof of Theorem 1

We use Proposition 9, the corresponding result with F replaced by F^{-1}, and the fact that $\sum_{\ell=1}^m \lambda_a^{(\ell)} = 0$ (because F is volume preserving). This gives an estimate of all the sums of $\lambda_a^{(\ell)}$ that occur in Theorem 1. ∎

11. Corollary

In the situation of Theorem 1, the central Lyapunov exponent is

$$\begin{aligned}\lambda^c &= \frac{a^2}{2}\sum_{-\infty}^{\infty}\int d\xi\,[\mathrm{Tr}^u(D_0^{-1}D'(\xi))\,\mathrm{Tr}^u(D_0^{-1}D'(F_0^n\xi)) \\ &\qquad\qquad - \mathrm{Tr}^s(D_0^{-1}D'(\xi))\,\mathrm{Tr}^s(D_0^{-1}D'(F_0^n\xi))] \\ &= \frac{a^2}{2}\sum_{-\infty}^{\infty}\int d\xi\,[\mathrm{Tr}^s(D_0^{-1}D'(\xi)) - \mathrm{Tr}^u(D_0^{-1}D'(\xi))]\,\mathrm{Tr}^c(D_0^{-1}D'(F_0^n\xi)),\end{aligned}$$

where Tr^s, Tr^u, Tr^c *denote the traces over the spectral subspaces* $\mathcal{E}^s$, $\mathcal{E}^u$, $\mathcal{E}^c$ *of* D_0 *corresponding to eigenvalues* < 1, > 1, *or* $= 1$ *in absolute value* ($\mathcal{E}^c$ *is one dimensional).*

Since F preserves the volume, the sum of all Lyapunov exponents vanishes. Therefore λ^c is minus the sum of the negative Lyapunov exponents, given by (5), minus the sum of the positive Lyapunov exponents. Note that replacing F by F^{-1}, $\mathcal{E}^s$ by $\mathcal{E}^u$ (and, to the order considered, $D'(\xi)$ by $-D'(\xi)$) replaces the sum of the negative Lyapunov exponents by minus the sum of the positive exponents. This gives the first formula for λ^c.

To obtain the second formula, express $\mathrm{Tr}^u\,\mathrm{Tr}^u - \mathrm{Tr}^s\,\mathrm{Tr}^s$ in terms of $\mathrm{Tr}^u \pm \mathrm{Tr}^s$, and remember that (because F preserves the volume) $\mathrm{Tr}^s + \mathrm{Tr}^u + \mathrm{Tr}^c = 0$ when applied to $D_0^{-1}D'(\xi)$. ∎

The above formula (5) takes a particularly simple form in a special case described in the next theorem.

12. Theorem

Let Φ *be a hyperbolic automorphism of* $\mathbf{T}^m$, *with stable and unstable dimensions* m^s *and* $m^u = m - m^s$, *and with entropy* λ_0^u. *Let* J: $y \to y + \alpha \pmod 1$ *be a translation of* $\mathbf{T}$, *and* ϕ: $\mathbf{T}^m \to \mathbf{T}$ *a group homomorphism* $\neq 0$. *Finally, let* ψ: $\mathbf{T} \to \mathbf{R}^m$ *be a nullhomotopic* C^2 *function.*

Define h, g_a: $\mathbf{T}^m \times \mathbf{T} \to \mathbf{T}^m \times \mathbf{T}$ *by*

$$h\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}\Phi x\\ Jy + \phi\Phi x - \phi x\end{pmatrix},\quad g_a\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x + a\psi(y) \pmod 1\\ y\end{pmatrix}$$

and let $f_a = g_a \circ h$.

Denote by λ_a^s *(resp.* λ_a^u*) the sum of the smallest* m^s *(resp. the largest* m^u*) Lyapunov exponents for* (f_a, volume). *Also let* $\lambda_a^c = -\lambda_a^s - \lambda_a^u$ *be the "central* *exponent". Then λ_a^s, λ_a^u, λ_a^c have expansions of order 2 in a:*

$$\lambda_a^s = -\lambda_0^u + \frac{a^2}{2}\int_{\mathbf{T}} dy\,((\nabla\phi)\psi'^s(y))^2 + o(a^2),$$
$$\lambda_a^u = \lambda_0^u - \frac{a^2}{2}\int_{\mathbf{T}} dy\,((\nabla\phi)\psi'^u(y))^2 + o(a^2),$$
$$\lambda_a^c = \frac{a^2}{2}\int_{\mathbf{T}} dy\,[((\nabla\phi)\psi'^u(y))^2 - ((\nabla\phi)\psi'^s(y))^2] + o(a^2).$$

Here, $\psi'^s(y)$ and $\psi'^u(y)$ are the components of the derivative $\psi'(y) \in \mathbf{R}^m$ in the stable and unstable subspaces $\mathcal{E}^s$ and $\mathcal{E}^u$ for Φ. Also, we have used $\nabla\phi\colon \mathbf{R}^m \to \mathbf{R}$ to denote the derivative of the map $\phi\colon \mathbf{T}^m \to \mathbf{T}$ with the obvious identifications.

This theorem is a simple (but nontrivial) extension of the result proved by Shub and Wilkinson [17]. In the situation that they consider $\Phi = \left(\begin{smallmatrix}2 & 1\\ 1 & 1\end{smallmatrix}\right)$, J =identity, $\phi = (1,0)$, $\psi' = \psi'^u$. [Remark that, in the notation of [17],

$$u_0 = ((1,1).v_0)/(m-1) = ((1,0).v_0),$$

so that the formula given in Proposition II of [17] agrees with our result above.]

Notation: We shall henceforth omit the (mod 1). We shall keep ∇ to denote the derivative in $\mathbf{T}^m$. With obvious abuses of notation, the reader may find it convenient to think of Φ or $\nabla\Phi$ as an $m \times m$ matrix (with integer entries and determinant ± 1), and ϕ or $\nabla\phi$ as a row m-vector (with integer entries not all zero).

13. Reformulation of the problem

Note that $f_a^{-1} = h^{-1} \circ g_a^{-1}$, where h^{-1}, g_a^{-1} are obtained from h, g_a by the replacements $\Phi, J, \phi, \psi \to \Phi^{-1}, J^{-1}, \phi, -\psi$. These replacements also interchange the stable and unstable subspaces for Φ and replace λ^s, λ^u by $-\lambda^u$, $-\lambda^s$. Therefore the formula for λ^u in the theorem follows from the formula for λ^s. And the formula for $\lambda^c = -\lambda^s - \lambda^u$ also follows. To complete the proof of the theorem we turn now to the formula for λ^s.

Define

$$\hat\phi\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}x\\ y+\phi x\end{pmatrix};$$

then

$$F_0\begin{pmatrix}x\\ y\end{pmatrix} = \hat\phi^{-1}h\hat\phi\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}\Phi x\\ Jy\end{pmatrix},$$

$$\hat{g}_a\begin{pmatrix}x\\y\end{pmatrix} = \hat{\phi}^{-1}g_a\hat{\phi}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x + a\psi(y+\phi x)\\ y - a(\nabla\phi)\psi(y+\phi x)\end{pmatrix}$$

so that

$$F\begin{pmatrix}x\\y\end{pmatrix} = \hat{\phi}^{-1}f_a\hat{\phi}\begin{pmatrix}x\\y\end{pmatrix} = \hat{g}_aF_0\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}\Phi x + a\psi(Jy+\phi\Phi x)\\ Jy - a(\nabla\phi)\psi(Jy+\phi\Phi x)\end{pmatrix}.$$

Finally, $F = F_0 + aF'$ with

$$F_0\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}\Phi x\\ Jy\end{pmatrix}, \quad F'\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}\psi(Jy+\phi\Phi x)\\ -(\nabla\phi)\psi(Jy+\phi\Phi x)\end{pmatrix}.$$

Since F is conjugate (linearly) to f_a, we may compute λ^s from F instead of f_a.

14. Proof of Theorem 12

Write $\mathbf{R}^{m+1} = \mathcal{E}^s + \mathcal{E}^u + \mathbf{R}$. We shall apply Proposition 9 with $\mathcal{E} = \mathcal{E}^s$, $\mathcal{E}^\perp = \mathcal{E}^u + \mathbf{R}$. Using $\xi = (x,y)$ and $X \in \mathcal{E}^s$, $Y \in \mathcal{E}^u$, $Z \in \mathbf{R}$ we may write

$$D_0\begin{pmatrix}X+Y\\ Z\end{pmatrix} = \begin{pmatrix}(\nabla\Phi)(X+Y)\\ Z\end{pmatrix},$$

$$D'(\xi)\begin{pmatrix}X+Y\\ Z\end{pmatrix} = \begin{pmatrix}\psi'(Jy+\phi\Phi x)((\nabla\phi\Phi)(X+Y)+Z)\\ -(\nabla\phi)\psi'(Jy+\phi\Phi x)((\nabla\phi\Phi)(X+Y)+Z)\end{pmatrix},$$

where ψ' denotes the derivative of ψ. Therefore

$$\mathrm{Tr}_{\mathcal{E}}(D'(\xi)D_0^{-1}) = (\nabla\phi)\psi'^s(Jy+\phi\Phi x)$$

and (5) contains the integrals

$$\int d\xi\, \mathrm{Tr}_{\mathcal{E}}(D_0^{-1}D'(\xi))\, \mathrm{Tr}_{\mathcal{E}}(D_0^{-1}D'(F_0^n\xi))$$

$$= \int d\xi[(\nabla\phi)\psi'^s(Jy+\phi\Phi x)][(\nabla\phi)\psi'^s(J^{n+1}y+\phi\Phi^{n+1}x)].$$

Performing a change of variables $\bar{x} = \Phi x$, $\bar{y} = Jy + \phi\Phi x$ we find that this is

$$= \int d\bar{x}d\bar{y}[(\nabla\phi)\psi'^s(\bar{y})][(\nabla\phi)\psi'^s(J^n\bar{y}+\phi\Phi^n\bar{x}-\phi\bar{x})].$$

We claim that this last integral vanishes unless $n = 0$. This is because, if $n \neq 0$,

$$\int d\bar{x}\psi'(J^n\bar{y}+\phi\Phi^n\bar{x}-\phi\bar{x}) = 0.$$

Indeed, $\phi\Phi^n\bar{x} - \phi\bar{x}$ is a linear combination with integer coefficients of the components $\bar{x}_1, \ldots, \bar{x}_m$ of $\bar{x}$, and the coefficients do not all vanish because $\phi\Phi^n = \phi$

is impossible (Φ is hyperbolic and $\phi \neq 0$). Integrating the derivative ψ' with respect to a variable $\bar{x}_j$ really occurring in $\phi\Phi^\ell\bar{x} - \phi\bar{x}$ gives zero as announced.

Returning to (5) we have thus

$$\begin{aligned}\lambda_a^s + \lambda_0^u &= \frac{a^2}{2}\int d\xi(\mathrm{Tr}_{\mathcal{E}}(D_0^{-1}D'(\xi)))^2\\ &= \frac{a^2}{2}\int d\bar{y}((\nabla\phi)\psi'^s(\bar{y}))^2,\end{aligned}$$

which is the formula given for λ_a^s in Theorem 12. And according to Section 13 this completes our proof. ∎

15. Final remarks

(a) Shub and Wilkinson [17] showed that close to a diffeomorphism (hyperbolic automorphism Φ of $\mathbf{T}^2$) $\times$ (identity on $\mathbf{T}$) there is a C^1 open set of ergodic volume preserving C^2 diffeomorphisms of $\mathbf{T}^3$ with central Lyapunov exponent $\lambda^c > 0$. They remark that their result extends to the situation where Φ is a hyperbolic automorphism of $\mathbf{T}^m$ with one-dimensional expanding eigenspace. More generally, if Φ is any hyperbolic automorphism of $\mathbf{T}^m$, Theorem 12 gives close to (Φ, rotation of $\mathbf{T}$) in $C^2(\mathbf{T}^{m+1})$ a diffeomorphism F with $\lambda^c > 0$. Since λ^c is given by an integral over the volume of a local "central" stretching exponent, we have $\lambda^c > 0$ in a C^1 neighborhood of F. But by a result of Dolgopyat and Wilkinson [8] (Corollary 0.5), stable ergodicity is here C^1 open and dense in the C^2 volume preserving diffeomorphisms (C^1 is improved to C^k in [12]): we have center bunching and stable dynamical coherence because we consider perturbations of (Φ, rotation of $\mathbf{T}$) for which the center foliation is C^1, see [10], [13]. In conclusion, close to (hyperbolic automorphism Φ of $\mathbf{T}^m$) $\times$ (rotation on $\mathbf{T}$) there is a C^1 open set V of ergodic volume preserving C^2 diffeomorphisms of $\mathbf{T}^{m+1}$ with central Lyapunov exponent $\lambda^c > 0$ (or also with $\lambda^c < 0$). In particular, if $F \in V$, the conditional measures of the volume on the circles $\pi^{-1}\{x\}$ are atomic, as discussed in [16].

(b) The coefficient L in Proposition 9 is ≥ 0. Consider indeed the unitary operator U defined by $U\psi = \psi \circ F$ on $L^2(\mathbf{T}^{m+1}, \text{volume})$, and let $E(.)$ be the corresponding spectral measure, so that

$$U = \int_{\mathbf{T}} e^{2\pi i\theta} E(d\theta).$$

If $\psi(\xi) = \mathrm{Tr}_{\mathcal{E}}(D_0^{-1}D'(\xi))$ we have a measure $\nu \geq 0$ on $\mathbf{T}$ defined by $\nu(d\theta) =$

$(\psi, E(d\theta)\psi)$ and the Fourier coefficients

$$c_n = \int e^{2\pi n i\theta}\nu(d\theta) = \int d\xi \operatorname{Tr}_{\mathcal{E}}(D_0^{-1}D'(\xi))(D_0^{-1}D'(F_0^n\xi))$$

of this measure tend to zero exponentially. Therefore $\nu(d\theta) = \rho(\theta)d\theta$ has a smooth density ρ and

$$L = \frac{1}{2}\sum_{n=-\infty}^{\infty} c_n = \frac{1}{2}\rho(0) \geq 0.$$

(c) Suppose now that F is not necessarily a volume preserving perturbation of F_0. We may still hope that F has an SRB measure ρ_a. If F_0 were hyperbolic, we would have an expansion

$$\rho_a = \rho_0 + a\delta + o(a)$$

(see [15]) with $\rho_0 =$ Lebesgue measure and δ a distribution. For smooth Ψ, $\delta(\Psi)$ is given (because ρ_0 is Lebesgue measure) by the simple formula (see [15])

$$\delta(\Psi) = -\sum_0^\infty \rho_0((\Psi \circ F_0^n).\operatorname{div}(F' \circ F_0^{-1})).$$

Similarly, replacing F by F^{-1}, hence F_0, $D_0^{-1}D'(\xi)$ by F_0^{-1}, $-D'(F_0^{-1}\xi)D_0^{-1}$, we see that the anti-SRB state has an expansion

$$\bar{\rho}_a = \rho_0 + a\bar{\delta} + o(a)$$

with

$$\begin{aligned}\bar{\delta}(\Psi) &= \sum_{n=1}^\infty \int d\xi \Psi(F_0^{-n}\xi) \operatorname{Tr}_{\mathbf{R}^{m+1}}(D'(F_0^{-1}\xi)D_0^{-1}) \\ &= \sum_{n=0}^\infty \int d\xi \Psi(F_0^{-n}\xi) \operatorname{Tr}_{\mathbf{R}^{m+1}}(D_0^{-1}D'(\xi)).\end{aligned}$$

We can now estimate the Lyapunov exponents for (F, ρ_a) to second order in a even though we are not sure of the existence of the SRB measure ρ_a. We simply assume that we can use the formula for $\delta(\Psi)$. Going through the proof of Proposition 9 we have to replace $\int d\xi \log N(\xi, a)$ by $\rho_a(\log N(., a))$ and (to second order in a) this adds to the right-hand side of (4) a term

$$-a^2 \sum_{n=1}^\infty \int d\xi \operatorname{Tr}_{\mathcal{E}}(D_0^{-1}D'(\xi)) \operatorname{Tr}_{\mathbf{R}^{m+1}}(D_0^{-1}D'(\xi)).$$

Taking into account the integrations by parts we obtain now instead of (5) the formula

$$L_a - (m_1\lambda_1 + \cdots + m_r\lambda_r) = \frac{a^2}{2} \sum_{n=-\infty}^{\infty} \int d\xi\, \mathrm{Tr}_{\mathcal{E}}(D_0^{-1}D'(\xi))\, \mathrm{Tr}_{\mathcal{E}}(D_0^{-1}D'(F_0^n\xi))$$

$$(6) \qquad - a^2 \sum_{n=-\infty}^{\infty} \int d\xi\, \mathrm{Tr}_{\mathcal{E}}(D_0^{-1}D'(\xi))\, \mathrm{Tr}_{\mathbf{R}^{m+1}}(D_0^{-1}D'(F_0^n\xi)).$$

Let a^2L^s, a^2L^u, a^2L^c be the a^2 contributions to the sum of the noncentral negative, noncentral positive, and the central Lyapunov exponents for the SRB measure. We obtain a^2L^s from (6) when $n_r = n^s$. A similar calculation gives a^2L^u (it is convenient here to work via the anti-SRB measure, then replace F by F^{-1}). Estimating the average expansion coefficient gives $a^2(L^s + L^u + L^c) = \rho_a(\log\det(D_0 + aD'(.))$, hence $L^s + L^u + L^c$, hence L^c. The results are

$$\begin{aligned}
L^s =& \frac{1}{2} \sum_{n=-\infty}^{\infty} \int d\xi\, \mathrm{Tr}^s(D_0^{-1}D'(\xi))\, \mathrm{Tr}^s(D_0^{-1}D'(F_0^n\xi)) \\
& - \sum_{n=-\infty}^{\infty} \int d\xi\, \mathrm{Tr}^s(D_0^{-1}D'(\xi))\, \mathrm{Tr}_{\mathbf{R}^{m+1}}(D_0^{-1}D'(F_0^n\xi)), \\
L^u =& -\frac{1}{2} \sum_{n=-\infty}^{\infty} \int d\xi\, \mathrm{Tr}^u(D_0^{-1}D'(\xi))\, \mathrm{Tr}^u(D_0^{-1}D'(F_0^n\xi)), \\
L^c =& -\frac{1}{2} \sum_{n=-\infty}^{\infty} \int d\xi\, \mathrm{Tr}^c(D_0^{-1}D'(\xi))\, \mathrm{Tr}^c(D_0^{-1}D'(F_0^n\xi)) \\
& - \sum_{n=-\infty}^{\infty} \int d\xi\, \mathrm{Tr}^c(D_0^{-1}D'(\xi))\, \mathrm{Tr}^u(D_0^{-1}D'(F_0^n\xi)), \\
L^s + L^u + L^c =& -\frac{1}{2} \sum_{n=-\infty}^{\infty} \int d\xi\, \mathrm{Tr}_{\mathbf{R}^{m+1}}(D_0^{-1}D'(\xi))\, \mathrm{Tr}_{\mathbf{R}^{m+1}}(D_0^{-1}D'(F_0^n\xi)),
\end{aligned}$$

which can be rewritten variously.

In view of recent work [4], [1], [6], it seems reasonable to conjecture that if the above L^c is $\neq 0$, then there exists an SRB measure for (small) finite a.

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References

[1] J. Alves, C. Bonatti and M. Viana, *SRB measures for partially hyperbolic systems whose central direction is mostly expanding*, Inventiones Mathematicae **140** (2000), 351–398.

[2] A. Avila and J. Bochi, *A formula with some applications to the theory of Lyapunov exponents*, Israel Journal of Mathematics **131** (2002), 125–137.

[3] C. Bonatti, X. Gómez-Mont and M. Viana, *Généricité d'exposants de Lyapunov non-nuls des produits déterministes de matrices*, Preprint.

[4] C. Bonatti and M. Viana, *SRB measures for partially hyperbolic systems whose central direction is mostly contracting*, Israel Journal of Mathematics **115** (2000), 157–193.

[5] K. Burns and A. Wilkinson, *Stable ergodicity of skew products*, Annales Scientifiques de l'École Normale Supérieure **32** (1999), 859–889.

[6] W. Cowieson and L.-S. Young, *SRB measures as zero-noise limits*, in preparation.

[7] D. Dolgopyat, *On differentiability of SRB states*, Preprint.

[8] D. Dolgopyat and A. Wilkinson, *Stable accessibility is C^1 dense*, Astérisque, to appear.

[9] L. Flaminio, *Local entropy rigidity for hyperbolic manifolds*, Communications in Analysis and Geometry **3** (1995), 555–596.

[10] M. Hirsch, C. Pugh, and M. Shub, *Invariant Manifolds*, Lecture Notes in Mathematics **583**, Springer, Berlin, 1977.

[11] A. Katok, G. Knieper and H. Weiss, *Formulas for the derivative and critical points of topological entropy for Anosov and geodesic flows*, Communications in Mathematical Physics **138** (1991), 19–31.

[12] V. Niţică and A. Török, *An open dense set of stably ergodic diffeomorphisms in a neighborhood of a non-ergodic one*, Preprint.

[13] C. Pugh and M. Shub, *Stable ergodicity and julienne quasi-conformality*, Journal of European Mathematical Society **2** (2000), 1–52.

[14] C. Pugh, M. Shub and A. Wilkinson, *Partial differentiability of invariant splittings*, Preprint.

[15] D. Ruelle, *Differentiation of SRB states*, Communications in Mathematical Physics **187** (1997), 227–241.

[16] D. Ruelle and A. Wilkinson, *Absolutely singular dynamical foliations*, Communications in Mathematical Physics **219** (2001), 481–487.

[17] M. Shub and A. Wilkinson, *Pathological foliations and removable exponents*, Inventiones Mathematicae **139** (2000), 495–508.